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The Cartan–Tresse linearization polynomial and applications

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Abstract

In this paper, we study the influence of the Cartan–Tresse linearization polynomial of a differential equation of order one in some classical problems of analysis, differential algebra and geometry, as singular solutions, the Ritt problem, or webs theory.

Keywords: Differential algebra; Singular solutions; Web geometry

1. Introduction

What is the subject of this article? In this article, we study first order differential equations of the form

$$P(x, y, y') = 0,$$

where $P \in K[y, y']$ is a polynomial, with coefficients in a field K containing $\mathbf{C}(x)$. This topic is naturally related to analysis, differential algebra and webs theory.

The Cartan–Tresse linearization polynomial of such differential equations naturally links up those subjects. Associated with any differential equation $P(x, y, y') = 0$ of order one, and originally defined with geometric technics and for geometric purposes, the Cartan–Tresse lineariza-

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tion polynomial V_P, or simply linearization polynomial, is a differential polynomial, defined in order to determine whether P is *linearizable*. That is to say, if after a suitable holomorphic change of coordinates, the integral curves of the equation are (germs of) straight lines. This quality of the linearization polynomial linked with a first order equation appeared in [2], Section 3. Nonetheless, we can trace it back to Tresse or Cartan works on projective connections (for a modern treatment, see also [12]).

In this article, we extensively use the linearization polynomial, with no direct links to its prime goal and, finally, we illustrate its ubiquity in the study of differential equations of order one.

Actually, we show that the linearization polynomial appears surprisingly in many other problems, coming from analysis, differential algebra or geometry. Those facts were unknown and besides, we show how using this polynomial to give explicit, new, partial or total answers to these problems. The use of the linearization polynomial offers a wide range of results, from the question of singular solutions of first order differential equations, of the Ritt problem, of the Ritt low power theorem to a Painlevé type property of such equations and, finally, to the study of planar webs.

What kind of results do we get? First of all, we give, in this article, an intrinsic definition of the linearization polynomial, *via* a Bézout type relation in the formalism of differential algebra (see Section 3.2, Theorem 2).

Mainly, the results that we get in this article can heuristically be described this way: we give different interpretations and consequences from the fact that the resultant $R_P := Result(P, \partial_{y'} P)$ "divides" V_P, in terms of remarkable properties of P. More generally, the relations between V_P and R_P are studied (see Section 4, Theorems 4, 5, Propositions 5 and 6).

We can sum up these results along two directions.

- ***Differential algebra and analysis***:

  - *Singular solutions and the Ritt problem*. We prove and study conditions for the existence of singular solutions of first order differential equations, via resultant and linearization polynomial (see Sections 4.1, 4.2, Proposition 3, Theorem 4, and Proposition 4). Surprisingly, the question of the existence of singular solutions appears related to the condition "R_P divides V_P." Finally, under this last condition, we give an explicit presentation for the general solutions of first order differential equations (see Section 4.3, Theorem 5).
  - *Painlevé type property*. We give an interpretation of the condition R_P divides V_P in terms of existence of poles of the derivation D, naturally induced by the datum of the differential equation $P(y, y') = 0$ (see Section 4.4, Propositions 5 and 6). Then, we get a relation between singular solutions and a type Painlevé type property for P (see Section 4.4, Corollary 5).

- ***Webs theory and geometry***:

  - *Poles of the connection*. Since the linearization polynomial determines the meromorphic connection associated with webs (see Section 5.2, Theorem 6), with poles on R_P, it is natural to study the influence of the property R_P divides V_P in this context. We partially characterize the poles of the determinant of the connection in terms of the condition R_P divides V_P (see Section 5.2, Theorem 7) and give a sufficient condition for the residue of the web to be 0 (see Section 5.2, Proposition 10).

2. Notations

• K will denote one of the following differential field: the field of fraction *Frac*($\mathbf{C}\{x\}$) of the ring of holomorphic functions, or the field of rational functions $\mathbf{C}(x)$, or $\mathbf{C}((x))$, the field of formal Laurent series, endowed with the usual derivation d/dx (that we will denote by δ). We fix an algebraic closure $\overline{K}$ of K, endowed with the unique derivation extending δ.

• One defines a unique derivation on the ring $K[y_i]$ of polynomials with an infinity of variables $y_0, y_1, y_2, \ldots$ by the adjunction of differential indeterminates

$$\delta(y_i) := y_{i+1},$$

for all i. Set $y := y_0$, we denote by $K\{y\}$ the δ-field obtained and we let $y_1 := y'$ and $y_2 := y''$.

A *δ-ideal of $K\{y\}$* is an ideal of $K\{y\}$ stable under the derivation δ.

Let S be a set in $K\{y\}$. We will respectively denote by (S), $[S]$ and $\{S\}$ the ideal, the δ-ideal, and the perfect δ-ideal (i.e. equal to its radical) generated by S. One can show that

$$\{S\} = \sqrt{[S]}.$$

• Let $0 \neq Q \in K[y, y']$, $R_\alpha \in K[y]$ be polynomials, such that R_α divides Q. The *multiplicity*, $\mathrm{mult}_Q(R_\alpha)$, of R_α in Q is the largest integer n such that $R_\alpha^n | Q$.

• If $P \in K\{y\}$ is a differential polynomial of order one, we take the following convention to see P as a polynomial in y' with coefficients in $K[y]$ and write it as the sum

$$P(y, y') := \sum_{i=0}^{d} a_{d-i}(y)(y')^i.$$

We call *degree* of such differential polynomial P the degree in y' of P (viewed as a polynomial with coefficients in $K[y]$).

A differential polynomial (of order one) P is said to be *irreducible*, if it is irreducible as a polynomial in $K[y, y']$. It is said to be *regular* if it does not contain a factor of degree zero.

We call *separant* of P, denoted S_P the derivative of P with respect to y', that is to say,

$$S_p := \frac{\partial P}{\partial y'}.$$

The quotient ideal of $\{P\}$ by S_P is defined to be

$$\bigl(\{P\} : S_P\bigr) := \bigl\{B \in K\{y\} \bigm| B \cdot S_P \in \{P\}\bigr\}.$$

We denote by R_P (or simply R) the y'-resultant $Result_{y'}(P, S_P)$ of P and S_P.

In the whole article, the polynomials $P \in K\{y\}$, that we consider, are always supposed to be without *multiple factors, and* not *of degree zero.*

3. Preliminaries

3.1. Singular solutions

In the analytic setting, a singular solution of a first order differential equation (or δ-polynomial) is a solution $y(x) \in L$ of the following differential system, where $L \supset K$ is an extension of K:

$$\begin{cases} P\big(x, y(x), y'(x)\big) = 0, \\ S_P\big(x, y(x), y'(x)\big) = 0. \end{cases}$$

Remark 3.1. In general, such singular solutions do not exist. The equation $(y')^2 + x = 0$, for example, does not admit a singular solutions.

In the differential algebraic setting, via Ritt decomposition theorem, one obtains the corresponding definition (see [6, Lemma 7.7]):

Theorem 1. *Let P be δ-polynomial in $K\{y\}$. If*

$$\{P, S_P\} = \bigcap_{t \in T} I_t$$

is the minimal decomposition in primes differential ideals of $\{P, S_P\}$, then $\{P\}$ admits the following decomposition:

$$\{P\} = \big(\{P\} : S_P\big) \cap \left(\bigcap_{s \in S} I_s\right)$$

where $S \subset T$ is the greatest set of indices such that for $s \in S$, the ideal I_s does not contain the ideal $(\{P\} : S_P)$.

A component of the preceding decomposition is called essential.

Definition 1.

(1) $(\{P\} : S_P)$ is the *component of the general solutions* of $\{P\}$.
(2) For $t \in T$, the ideal I_t is a *component of the singular solutions*.
(3) The ideals I_s, for $s \in S$, are the *essential* components of the singular solutions.
(4) The ideals I_p, for $p \in T - S$ are the *particular* components of the singular solutions.

Remark 3.2. One must be careful with these definitions. Consider the differential equation:

$$P(y, y') = y(y')^3 + y^2 = 0.$$

It is easy to see that the prime differential ideal $\{y\}$ is an *essential* component of the minimal decomposition of $\{P\}$. It is also a *component of the singular solutions*. But it is a *particular* component of the singular solutions, not an *essential* component of the singular solutions for $\{P\}$. That is to say, $\{y\}$ is a component of the general solutions of $\{P\}$.

3.2. The Cartan–Tresse linearization polynomial

In this section, we give an intrinsic definition of the linearization polynomial. The following theorem is a very important technical point in this article. This is a more general version of a theorem proved in [15, Proposition 8] or [14, Theorem 3.1].

Theorem 2. *Let $L \supset K$ be an algebraic extension of K and $P(x, y, y') := a_0(y')^d + a_0(y')^{d-1} + \cdots + a_d = 0$ a differential polynomial in $L\{y\}$ of order one and degree $d \geqslant 2$ in y'. There exists two differential polynomials U_P and V_P in $L\{y\}$ of order one and respective degree at most $d-2$ (or 1 if $d = 2$) and $d-1$, such that the following equality holds*:

$$(\diamond) \quad R(\partial_x P + y'\partial_y P) = U_P \cdot P + V_P \cdot \partial_{y'}(P)$$

where $R := Res(P, \partial_{y'}(P))$ if $d \geqslant 3$, and $R := a_0 \cdot Res(P, \partial_{y'}(P))$ if $d = 2$.

Moreover, such an expression is unique: if two polynomials $\overline{U}$ and $\overline{V}$ in y' respectively of degree at most $d-2$ (or 1 if $d = 2$) and $d-1$ and $\overline{R}$ of degree 0 satisfy $(\diamond)$, then $R \cdot \overline{U} = \overline{R} \cdot U_P$ and $R \cdot \overline{V} = \overline{R} \cdot V_P$.

Proof. We follow the proof of [15, Proposition 8], but in a more general setting. If $d \geqslant 3$, such polynomials $U_P := u_2 \cdot p^{d-2} + \cdots + u_d$ and $V_P := v_1 \cdot p^{d-1} + \cdots + v_d$ must satisfy a system $S(\diamond)$, deduced from $(\diamond)$. If $\mathcal{R}$ stands for the Sylvester square matrix of order $2d-1$ whose determinant is R, the system is given by

$$\begin{aligned} S(\diamond) \quad & \mathcal{R} \cdot {}^t(u_2, \ldots, u_d, v_1, \ldots, v_d) \\ & = {}^t\big(0, \ldots, 0, \partial_y(a_0), \partial_x(a_0) + \partial_y(a_1), \ldots, \partial_x(a_{i-1}) + \partial_y(a_i), \ldots, \partial_x(a_d)\big). \end{aligned}$$

We then get $(\diamond)$ and its uniqueness *via* Cramer's rule. If $d = 2$, the same proof holds with $R = a_0 \cdot \mathrm{Res}(P, \partial_{y'}(P))$. $\square$

Remark 3.3. In particular, it is meaningful to define U_P and V_P over $\overline{K}$.

Definition 2 *(Cartan–Tresse linearization polynomial).* The differential polynomial V_P of order one and degree at most $d-1$ in $\overline{K}[y, y']$ (that we denote simply by V if no confusions are possible) defined in the previous theorem will be called the *Cartan–Tresse linearization polynomial* or *linearization polynomial* associated with $P \in \overline{K}[y, y']$.

The following proposition clearly establishes the link between the usual definition of the linearization polynomial and ours. It was proved in [15, Théorème 11].

Proposition 1. *With the previous notations,*

(1) *$R_P y'' + V_P \in (\{P\} : S_P)$, that is to say that the general solutions of P is also solution of the second order equation $R_P y'' + V_P = 0$;*
(2) *The general solutions of P are (germ of) straight lines if and only if $V_P = 0$.*

4. Singular solutions, general solutions, and Painlevé properties

In this paragraph, the polynomials $P \in K\{y\}$, that we consider, are always supposed to be *without* multiple factors, and *not* of degree zero.

4.1. Some remarks on singular solutions

In [15, Proposition 1], we have established, in the differential algebraic setting, the following link between singular solutions and the resultant of an algebraic differential equation of order one.

Proposition 2. *Let $P(x, y, y') = 0$ be a regular differential equation of order one and degree d in y' and I a component of the singular solutions of $\{P\}$. There exists then an irreducible factor R_s of the resultant $Res(P, \partial_{y'}(P))$ such that $I = \{R_s\}$.*

We can easily deduce the following corollary:

Corollary 1. *Let P be a differential polynomial of order one in $K\{y\}$. Each singular solution of P is algebraic over K.*

Example 4.1. Let P be a differential polynomial of order one in $K\{y\}$, such that $\partial_x P = 0$. If $\{P\}$ admits a component of the singular solutions, then this component is of the form $\{y - a\}$ where $a \in \mathbf{C}$. Indeed, such a component is given by an irreducible factor of the resultant. But, since $\partial_x P = 0$, the resultant is in $\mathbf{C}[y]$, and so each of its irreducible factors are of the form $(y - a)$, where $a \in \mathbf{C}$, since $\mathbf{C}$ is algebraically closed.

Example 4.2. Let $d \geqslant 2$ be an integer. Consider the Malmquist polynomial

$$P(x, y, y') := a_0(y)(y')^d + a_d(y)$$

such that the polynomials $(0 \neq)$ $a_i \in K[y]$ are irreducible and relatively prime. The y'-resultant of such P is given by the formula $R = (da_0)^d a_d^{d-1}$. Then, following Proposition 2, it is easy to show there is at most one component of the singular solutions, which is of the form $\{y - a\}$, $a \in \mathbf{C}$. Indeed, by assumptions, we can prove that $y' \in \{a_d\}$. By the formula

$$\delta(a_d) = \partial_x(a_d) + y'\partial_y(a_d),$$

we conclude that a_d divides $\partial_x(a_d)$. So, $a_d = \lambda\tilde{a}_d$, where $\lambda \in K$ and $\tilde{a}_d \in \mathbf{C}[y]$. The conclusion comes from the same arguments given in Example 4.1.

Remark 4.1. If $\eta(x) \in \overline{K}$ is a singular solution of a differential equation of first order such that R has no multiple factors, it is easy to show that η is solution of a particular, explicit Cauchy–Lipschitz differential equation of first order. Indeed, if $1 = uR + vS_R$ is the identity of Bézout, since $\delta(R) = \partial_x(R) + y'S_R$, we have $u\delta(R) = v\partial_x(R) + y'(1 - uR)$. And η is then solution of the following differential equation:

$$y' + v\partial_x(R) = 0.$$

The low power theorem of Ritt gives an effective way to find essential components of a differential polynomial (see [7, Exercise 1, Section IV/16, page 193], for a formulation in the first order case). We give here a very useful proposition, which can also be seen as a little more direct formulation of this statement in the first order case, when $\{y\}$ is the component of singular solution (still, in the study of singular solutions, it is legitimate to make this assumption).

Proposition 3. *Let $P = a_0(y')^d + a_1(y')^{d-1} + \cdots + a_d$ in $K\{y\}$ a differential polynomial over K, of order one and degree d. Let m_i be the multiplicity of y in a_i, when $a_i \neq 0$. We suppose that $a_d \neq 0$. Then $\{y\}$ is an essential component of the minimal decomposition of $\{P\}$ if and only if $m_d \geqslant 1$ and*

$$m_i \geqslant m_d - (d-i) + 1.$$

Proof. Since $y = 0$ is a solution of $P = 0$, we have $m_d \geqslant 1$. Moreover, P is a preparation polynomial with regard to y. By the low power theorem (see [16, Chapter III, Section 20]) the component $\{y\}$ is essential if and only if a_d is of non-zero lower multidegree in y (and y') than every other terms in P. That is to say that $m_i + d - i > m_d$.

We also can give a geometric meaning of this criterion. Take P as in Proposition 3. By the criterion above, we can write:

$$\frac{P(y,y')}{y^d} = \frac{1}{y^d}\sum_{i=0}^{d} y^{m_{d-i}}\tilde{a}_{d-i}(y')^i$$

where $a_{n-i} := y^{m_{d-i}} \cdot \tilde{a}_{n-i}$, and $\tilde{a}_{n-i} \in K[y]$. So,

$$\frac{P(y,y')}{y^d} = \sum_{i=0}^{d} y^{m_{d-i}-(d-i)}\tilde{a}_{d-i}\left(\frac{y'}{y}\right)^i.$$

It follows that $\{y\}$ is an essential component of the decomposition of the ideal $\{P\}$ if and only if

$$P(y,y') = y^{m_d}\sum_{i=0}^{d} y^{((m_{d-i}-(d-i))-(m_d-d))}\tilde{a}_{d-i}\left(\frac{y'}{y}\right)^i$$

and, for all $d > i \geqslant 0$, $m_i - i > m_d - d$ (for indices i such that $a_i \neq 0$). Putting $yt := y'$, we obtain the following statement:

Corollary 2. *Let $P = a_0(y')^d + a_1(y')^{d-1} + \cdots + a_d$ in $K\{y\}$ a differential polynomial over K, of order one and degree d. We suppose that $a_d \neq 0$. Then $\{y\}$ is an essential component of $\{P\}$ if and only if there exists $G(y,t) = \sum_{i=0}^{d} g_{d-i}(y)t^i$ in $K[y,t]$ and $n \in \mathbf{N}\backslash\{0\}$, such that $G(y,0) := g_d(y) \in K\backslash\{0\}$, $\mathrm{mult}_{g_i}(y) > 1$ for all $d > i \geqslant 0$, and*

$$P(y,yt) = y^n \cdot G(y,t).$$

In this case, $n = \mathrm{mult}_{a_d}(y)$.

The following statement is a more direct formulation in the context of singular solutions, and is a generalization (and so gives a proof) of Lemma 1 in [9]:

Corollary 3. *let P be a differential polynomial of order one in $K\{y\}$, such that y does not divide P. Then $\{y\}$ is an essential component of the singular solutions if and only if $m_d \geqslant 1$ and*

$$m_i \geqslant m_d - (d-i) + 1.$$

Proof. From Proposition 3, we only have to show that the valuative criterion implies that $\{y\}$ is a singular solution. If it is not the case, the essential component $\{y\}$ is a component of the general solution. If $P = \prod_i P_i$, with $P_i \wedge P_j = 1$ if $i \neq j$, is the decomposition in irreducible factors P_i of P, then we can show (see [15, Proposition 3]) that $(\{P\} : S_P) = \bigcap_i (\{P_i\} : S_{P_i})$. So, there exists i such that $\{y\} = (\{P_i\} : S_{P_i})$. Since P_i is irreducible and its order is 1, we have $P_i = y$, which is excluded by hypothesis.

Remark 4.2. If $\eta(x) \in K$ is an integral solution of P, the criterion of Corollary 3 gives an effective test in order to check if $\eta(x)$ is an essential singular solution, or not, for first order differential equations (by making the change of variables defined by $u := y - \eta(x)$). Equivalently, it allows to check if this solution $\eta(x)$ is, or not, a general solution of the equation $P = 0$ (this is the Ritt problem, see Section 4.3).

4.2. Singular solutions and the Cartan–Tresse linearization polynomial

The following theorem on singular solutions involves the linearization polynomial and complete the statement of [15, Théorème 12]:

Theorem 3. *Let $P(x, y, y') = 0$ be a regular differential equation of order one and degree d. Let $R := Result_{y'}(P, S_P)$. If $\{P\}$ admits a component $\{R_s\}$ of the singular solutions, where R_s is an irreducible factor of the resultant R, then $V \in \{R_s\}$.*

Proof. For an essential component $\{R_s\}$ of the singular solutions, see [15, Théorème 12]. In the case of a particular component $\{R_s\}$ of the singular solutions, it is sufficient to remark that $Ry'' + V \in \{R_s\}$. So, in this case, V belongs to $\{R_s\}$ too. □

This result is not optimal as we can see with the lemma below.

Lemma 1. *Let $P \in K\{y\}$ be an irreducible differential polynomial of order one and degree d, V its linearization polynomial, and $R := Result_{y'}(P, S_P)$ denotes its resultant. Let R_s be an irreducible factor of R. Then, if the degree of V is 0, then R_s divide V.*

Proof. The order of R_s, which is irreducible by assumption, is 0. So, since $V \in \{R_s\}$, then R_s divides V. □

Here, we want to complete this result. First, let us define the *rank of an irreducible factor of the resultant*.

Definition 3. Let $P \in K\{y\}$ be a differential polynomial of order one and degree d, $V \neq 0$ its linearization polynomial, and $R := Result_{y'}(P, S_P)$ denotes its resultant. Let R_s be an irreducible factor of R. We call the rank of $\{R_s\}$ in V the integer

$$rk_V(R_s) := \operatorname{mult}_R(R_s) - \operatorname{mult}_V(R_s).$$

Theorem 4. *Let $P \in K\{y\}$ a differential polynomial over K of order one and degree d, $R := Result_{y'}(P, S_P)$ its resultant and V its linearization polynomial. Suppose that $\{R_s\}$ is a component of the singular solutions of $\{P\}$, where R_s an irreducible factor of R. Then R_s divides V.*

In other words, whether $V = 0$, whether the rank of $\{R_s\}$ satisfies the inequalities:

$$rk_V(R_s) \leqslant \operatorname{mult}_R(R_s) - 1.$$

To prove this theorem, we need the following lemma:

Lemma 2. *Let $P \in \overline{K}\{y\}$ be a differential polynomial of order one and degree d, V its linearization polynomial, and $R := Result_{y'}(P, S_P)$ denotes its resultant. Let $R_s := y - \eta(x)$, $\eta(x) \in \overline{K}$, be an irreducible factor of R. Then, if $T(u, v) := P(x, u + \eta, v + \eta')$, we have the formula*

$$V_T(x, u, v) = V_P(x, u + \eta, v + \eta') + \eta'' R(x, u + \eta).$$

Proof. The relation $(\diamond)$ for T is

$$R_T\big(\partial_x(T) + v\partial_u(T)\big) = U_T \cdot T + V_T \cdot \partial_v(T)$$

where R_T is the v-resultant of T and $S_T := \partial_v(T)$. Since $S_T = S_P(x, u + \eta, v + \eta')$, the relation $(\diamond)$ written for P, applied in $(x, u + \eta, v + \eta')$ gives that

$$\begin{aligned}
&R(x, u + \eta)\big(\partial_x(P)(x, u + \eta, v + \eta') + v\partial_y(P)(x, u + \eta, v + \eta')\big)\\
&\quad = U(x, u + \eta, v + \eta') \cdot P(x, u + \eta, v + \eta') + \big(V(x, u + \eta, v + \eta') + R_T\eta''\big)\\
&\qquad \cdot \partial_{y'}(P)(x, u + \eta, v + \eta').
\end{aligned}$$

Thus by the uniqueness of the relation $(\diamond)$, we have the equality

$$R(x, u + \eta) \cdot V_T = R_T.\big(V(x, u + \eta, v + \eta') + R(x, u + \eta)\eta''\big).$$

Since the change of coordinates is a translation, we have $R_T(x, u) = R(x, u + \eta)$. Indeed, it is enough to consider the definition of the resultant as the product of the roots of P in $\overline{K}$. The fact that η is a root of P allows us to conclude. $\square$

Proof of Theorem 4. We can view P in $\overline{K}\{y\}$, and so suppose that $R_s = y - \eta$, with $\eta \in \overline{K}$. Then, we only have to show that $y - \eta$ divides V in $\overline{K}[y, y']$. Let $u := y - \eta$ and $v := y' - \eta'$. We consider the polynomial $T(u, v)$ in $K[u, v]$ given by $T(u, v) := P(x, u + \eta, v + \eta')$. By Lemma 2, it is sufficient to prove that u divides V_T.

So, we suppose, now, that $R_s = y$ and $P \in K\{y\}$. We have to show that y divides V. Let us write $P(x, y, y') = \sum_{i=1}^{d} a_i(x, y)(y')^{d-i}$. Thus, $y = 0$ is a solution of $P = 0$ and $S_P = 0$, and so, the order m_d and m_{d-1} of y in a_d and a_{d-1} are greater or equal to 1. But if we are to compute V, we find (see [14, Theorem 3.1] or [15, Proposition 8]) that the coefficients v_i, for $d \geqslant i \geqslant 1$, of V are, thanks to Cramer's rule, determinants whose last line are of the form

$$\big(0 \; \ldots \; 0 \; a_d \; 0 \; \ldots \; \partial_x(a_d) \; 0 \; \ldots \; 0 \; a_{d-1}\big)$$

if $d-1 \geqslant i \geqslant 1$, and where the coefficient $\partial_x(a_d)$ belongs to the $d-1+i$ column. When $i=d$, the line is of form:

$$\big(0 \; \ldots \; 0 \; a_d \; 0 \; \ldots \; 0 \; 0 \; \ldots \; 0 \; \partial_x(a_d)\big).$$

This shows that V is of order at least 1 in y, and thus y divides $V(x, y, y')$. That is to say that R_s divides V. $\square$

Generally, R does not divide V, as one can see it in the following examples:

Example 4.3. Consider $P = (y')^2 - y + x^2$ in $K[y, y']$. In this case $R = 4(x^2 - y)$ is irreducible and $V = -4xy' - 2(x^2 - y)$. There is no singular solutions.

Example 4.4. Let us consider the equation $(y')^3 + y^2 = 0$. Then $R = 27y^4$. Since $V = -2y^3/3$, the component $\{y\}$ is of rank 1. In fact, on can show that it is an essential component of the singular solutions of this equation.

Example 4.5. The equation $(y')^3 - 4xyy' + 8y^2 = 0$ admits the resultant $R = -64y^3(4x^3 - 27y)$. Since $V = 32y^2(4x^3 - 27y)$, the component $\{y\}$ is of rank 1, but it is a particular component of the singular solutions. The component $\{4x^3 - 27y\}$ is of rank 0 and is essential.

Example 4.6. The equation $(y')^3 + x(y')^2 - y = 0$ admits the resultant $R = y(27y - 4x^3)$, and the linearization polynomial $V = y(-3(2x+3)(y')^2 - x(2x+3)(y') + (2x^2 + 9y))$. In this case, R does not divide V, and this equation admits $\{y\}$ as an essential component of singular solutions.

Example 4.7. The Malmquist polynomials of the form $P(x, y, y') := a_0(y')^d + a_d$ of Example 4.2, with $a_i \in K[y]$ irreducible and relatively prime, admits the y'-resultant $R = (da_0)^d a_d^{d-1}$ of P. Its linearization polynomial is

$$V := d^{d-1} \cdot a_d^{d-2} \cdot a_0^{d+1} \cdot \left(\partial_y\left(\frac{a_d}{a_0}\right) \cdot (y')^2 + \partial_x\left(\frac{a_d}{a_0}\right) \cdot y'\right).$$

Assume $a_0 = 1$. Since a_d is a polynomial in y, we see that R divides V if and only if $\partial_y(a_d) = 0$ (and $a_d | \partial_x(a_d)$).

Corollary 4. *If R is irreducible, and P, $S_P \in \{R\}$, then R divides V.*

Proposition 4. *Let P be a regular differential equation of order one. Let us suppose that $R := \mathrm{Result}_{y'}(P, S_P)$ is irreducible. Assume $\partial_y(P) \notin \{P, S_P\}$, then $\{R\}$ is an essential component of the singular solutions of $\{P\}$ if and only if $R | V$.*

Proof. The direct implication is a consequence of Corollary 4. The other implication has been proved in [15, Théorème 8]. □

4.3. The Ritt problem and the Cartan–Tresse linearization polynomial

A complementary question to the one on singular solutions of differential equations is the question of general solutions. The problem of "recognizing" general solutions is called the Ritt problem (see [7, Section IV/16]).

The statement of the following Theorem 5 gives a direct and very explicit answer to the Ritt problem for first order differential equation, in the case where $R|V$.

Example 4.8. The condition R divides V occurs*:* consider for example

$$P := -x^3(y')^3 + 3\big(y - x^4\big)x^2(y')^2 - \big(3\big(y - x^4\big)^2 x - 1\big)y' + \big(y - x^4\big)^3 + \frac{4}{3}x^3.$$

We have $R = -x^6(3x^9 + 27xy^2 + 18yx^5 - 4)$ and $V = 4x^2 R$. Here, we see that R divides V.

Theorem 5. *Let P be an irreducible differential polynomial of order one, with $(0 \neq)R :=$ $\mathrm{Result}_{y'}(P, S_P)$ and V its linearization polynomial. Suppose that R divides V. Then*

$$\big(\{P\} : S_P\big) = \left\{P, y'' + \frac{V}{R}\right\}.$$

Remark 4.3. The condition $R|V$ is a natural one (see Theorem 4 and Proposition 4 above), sufficient to establish the existence of essential singular solutions (without such an existence, the Ritt problem is, of course, trivial). Besides, our statement does not use the Ritt low power theorem, and gives an interpretation of the Ritt problem via the linearization polynomial.

Proof. Denote $Q := \partial_x P + y'\partial_y P$. By Theorem 2, we have $RQ = U \cdot P + V \cdot S_P$. Since P is irreducible, R divides U. Let $\widetilde{U} = U/R$ and $\widetilde{V} = V/R$ in $K[y, y']$, so that $Q = \widetilde{U} \cdot P + \widetilde{V} \cdot S_P$. We have $\delta(P) = Q + y''S_P$ so $S_P \cdot (\widetilde{V} + y'') = \delta(P) - \widetilde{U} \cdot P$. Thus,

$$\{P, y'' + \widetilde{V}\} \subseteq \big(\{P\} : S_P\big).$$

Using the relation $y'' + \widetilde{V} = 0$, we can find, for each i, a differential polynomial $\widetilde{V_i}$ of order less or equal to 1 such that $y^{(i)} - \widetilde{V_i} \equiv 0 \bmod [y'' + \widetilde{V}]$ (here $\widetilde{V_0} := -\widetilde{V}$). As a first consequence, notice that $K\{y\}/[P, y'' + \widetilde{V}] \simeq K[y, y']/P$ is a domain, so the ideal $[P, y'' + \widetilde{V}] = \{P, y'' + \widetilde{V}\}$ is prime. Let B be in $(\{P\} : S_P)$ of order n. Let $\widetilde{B} := B(x, y, y', \widetilde{V_0}, \widetilde{V_1}, \dots, \widetilde{V_n})$. Then $B \equiv \widetilde{B} \equiv 0 \bmod (\{P\} : S_P)$. But the order of $\widetilde{B}$ is less or equal to 1, and $\widetilde{B} \in (\{P\} : S_P)$. Since P is irreducible and of order one, then P divides $\widetilde{B}$ (see [1, Chapter 2, Theorem 2.4]). So $\widetilde{B} = ZP$ in $K\{y\}$, but $B = \widetilde{B} \bmod [y'' + \widetilde{V}]$. So $B = ZP \bmod [y'' + \widetilde{V}]$ and $B = 0 \bmod [P, y'' + \widetilde{V}]$, which proves that $B \in \{P, y'' + \widetilde{V}\}$, and the requested equality. □

Example 4.9. The following example can be found in [16, Example 1, page 120]. Let $P :=$ $(y')^2 - 4y$. By Corollary 3, we know that $\{y\}$ is a component of the singular solutions of $\{P\}$. In this case, $R = -16y$ and $V = 32y$. Then, by Theorem 5, we can conclude that the general solutions are determined by the prime differential ideal $\{(y')^2 - 4y, y'' - 2\}$. This is the result obtained by Ritt.

Example 4.10. Consider (polynomial) Clairaut equations given by the generic formula $P := g(y') + xy' - y$, with $g(t) \in K[t]$. In this case, $V = 0$ (see [15, Proposition 5]). So, by Theorem 5, general solutions are given by the prime differential ideal $\{P, y''\}$. In [8] (see example page 575), A. Buium and P.J. Cassidy obtain this result in the particular case of the equation $P = (y')^2/4 - xy' + y$ (which is a Clairaut equation with $g(t) = -t^2/4$).

4.4. *Painlevé type property and the Cartan–Tresse linearization polynomial*

Let $P(y, y') = 0$ be an irreducible differential equation of order one in $K[y, y']$. Let $X \hookrightarrow \mathbb{A}^2_K$ be the integral curve defined by $P(s,t) = 0$ and $K(P)$ be its functions field. Let V be the linearization polynomial of P, and $(0 \neq) R := \mathit{Result}_{y'}(P, S_P)$. We suppose that $(P, R) \neq K[s,t]$.

In this paragraph, we will realize the condition "R divides V" as a differential relation of Painlevé type.

Let D be the derivation defined by $D(s) := t$ on $K(P)$. Thus, $(K(P), D)$ is a differential field.

Definition 4. We say that a closed point x in X is a *local pole of D* if

$$D(\mathcal{O}_{X,x}) \nsubseteq \mathcal{O}_{X,x}$$

in $K(P)$.

Remark 4.4. A "global" version of this condition is related to the famous Fuchs conditions for ordinary differential equations of order one (see [10], remark page 14), when X is regular.

Via the linearization polynomial V, we can compute $D(t)$ in $K(P)$. By definition of D, we have

$$0 = D(P) = \partial_x(P) + t\partial_y(P) + D(t)S_P$$

in $K(P)$. But Theorem 2 gives us that $R(\partial_x(P) + t\partial_y(P)) = VS_P$, so that

$$R \cdot D(t) + V = 0 \quad \text{in } K(P).$$

As R is invertible in $K(P)$, we have

$$D(t) = -\frac{V}{R}.$$

Proposition 5. *The differential field $(K(P), D)$ has no local poles in X, then*

$$V \in \sqrt{(R, P)}.$$

Proof. Suppose that $V \notin \sqrt{(R, P)}$. There exists $\mathcal{Q}(R)$ a maximal ideal containing (R, P), which does not contain V. In this case, the ideal $\mathcal{Q}(R)$ is a local pole of D. Indeed, we have $t \in \mathcal{O}_{X,\mathcal{Q}(R)}$ since

$$t \in K[s,t]/P \hookrightarrow \mathcal{O}_{X,\mathcal{Q}(R)}.$$

If $D(t) \in \mathcal{O}_{X,\mathcal{Q}(R)}$, then there exists A and B in $K[s,t]/P$, with $B \notin \mathcal{Q}(R)$ so that $-V/R = A/B$. So, VB is in $\mathcal{Q}(R)$, and V is in $\mathcal{Q}(R)$, which is a contradiction. So $\mathcal{Q}(R)$ is a pole of $(K(P), D)$, which proves the proposition. □

Example 4.11. The following examples have been studied in [11]. We obtain here the existence of poles by Proposition 5 (see [11, Sections 1 and 4] for a direct computation).

- Consider the polynomial $P = (y')^2 - y + x^2$. Then $R = 4(x^2 - y)$ and $V = -4xy' + 2(y - x^2)$. In this case, R is irreducible and does not divide V. So there exists a (local) pole.
- Consider the polynomial $P = (y')^2 - y^3 - x$ in $\mathbf{C}(x)[y, y']$. Then $R = -(y^3 + x)$ and $V = 2y' + 6(y^3 + x)y^2$. In this case, R is irreducible and does not divide V. So there exists a (local) pole.

Proposition 6. *If $V \in (R, P)$, then $(K(P), D)$ has no local poles in X.*

Proof. By hypothesis, we have $V = RZ$ in $K[s,t]/P$. Then $D(t) = -Z$ in $K[s,t]/P$. Let $\mathcal{Q}$ be a closed point in X. Let $L := A/B$ in $\mathcal{O}_{X,\mathcal{Q}}$, with $A, B \in K[s,t]/P$ and $B \notin \mathcal{Q}$. By definition,

$$D(L) = \frac{B \cdot D(A) - A \cdot D(B)}{B^2}.$$

In order to prove that $D(L) \in \mathcal{O}_{X,\mathcal{Q}}$, it is enough to prove that if $M \in K[s,t]/P$, then $D(M) \in K[s,t]/P$. But we have $D(M) = \partial_x M + t\partial_y M + D(t)S_M$. Since $D(t) = -Z \in K[s,t]/P$, and $K[s,t]/P \hookrightarrow \mathcal{O}_{X,\mathcal{Q}}$ we have proved the requested assertion. So $D(\mathcal{O}_{X,\mathcal{Q}}) \subseteq \mathcal{O}_{X,\mathcal{Q}}$. □

By gluing these statements with the statement of Proposition 4, we get the following result:

Corollary 5. *Let P be a regular differential equation of order one. Let $R := Result_{y'}(P, S_P)$, that is supposed to be irreducible. Then D has no local poles, if $\{R\}$ is a component of the singular solutions of $\{P\}$.*

5. Webs and the Cartan–Tresse linearization polynomial

In this paragraph, the polynomials $P \in K\{y\}$ that we consider, are always supposed to be *without* multiple factors, and *not* of degree zero.

5.1. Planar webs

For a more detailed introduction to webs, one can refer to [17] or [5].

According to the classical definition, a d-web is given by a family of d foliations (the leaves of the web) defined by level sets $F_i(x, y) = \text{const}$ for $1 \leqslant i \leqslant d$ where the F_i are in $\mathcal{O} := \mathbf{C}\{x, y\}$ in general position, with $F_i(0) = 0$. Web geometry is devoted to the study of such configurations.

Equivalently, the d leaves of a non-singular planar d-web $\mathcal{W}(d)$ are given by the d integral curves of a differential equation of order one and degree d

$$P(x, y, y') := a_0(x, y) \cdot (y')^d + a_1(x, y) \cdot (y')^{d-1} + \cdots + a_d(x, y) = 0$$

which is a y'-polynomial with coefficients in $\mathcal{O}$, in a neighborhood of a point where the y'-resultant R_P is non-zero.

We say that a d-web is *presented* by such a differential equation. In the sequel, a d-web will always be presented by such a differential equation, denoted $P = 0$, if it is not otherwise specified.

Let us define the main invariant in web geometry. An abelian relation is then a relation of the form

$$\sum_{i=1}^{d} g_i(F_i)\, dF_i = 0,$$

where the g_i are analytic in one variable. The $\mathbf{C}$-vector space defined by

$$\mathcal{A}(d) = \left\{ \big(g_1(F_1), \ldots, g_d(F_d)\big) \in \mathcal{O}^d \text{ with } g_i \in \mathbf{C}\{t\} \text{ and } \sum_{i=1}^{d} g_i(F_i)\, dF_i = 0 \right\}$$

is called the space of *abelian relations* of the d-web $\mathcal{W}(d)$

The dimension of $\mathcal{A}(d)$ is finite and called the *rank of the web*. By a classical theorem in web geometry, we have the following optimal inequality:

$$rk\, \mathcal{W}(d) := \dim_{\mathbf{C}} \mathcal{A}(d) \leqslant \frac{1}{2}(d-1)(d-2).$$

5.2. *Some remarks on webs*

First of all, since a d-web is presented by a differential polynomial P of order one and degree d, up to an invertible in $Frac(\mathbf{C}\{x\})[y]$, we will talk about the linearization polynomial of $\mathcal{W}(d)$, meaning that we consider a regular presentation P of the web and the linearization polynomial V_P. This denomination is justified by the following lemma:

Lemma 3. *Let g be an invertible of $Frac(\mathbf{C}\{x\})[y]$. Then $V_{gP} = g^{2d-1} V_P$. Thus*

$$R_{gP} \cdot V_P = R_P \cdot V_{gP}.$$

Proof. Consider the relation $(\diamond)$ for gP. We have:

$$R_{gP}\big(\partial_x(gP) + y'\partial_y(gP)\big) = U_{gP} \cdot gP + V_{gP} \cdot \partial_{y'}(gP).$$

But $R_{gP} = g^{2d-1} R_P$ and $g^{2d-1} R_P(\partial_x(g) + y'\partial_y(g))P + g^{2d-1} R_P(\partial_x(P) + y'\partial_y(P)) = gU_{gP} \cdot P + V_{gP} \cdot g\partial_{y'}(P)$. The uniqueness of the relation $(\diamond)$ applies and we have $V_{gP} = g^{2d-1} V_P$ $\square$

We are now interested in special webs, given by a $d-1$ algebraic web, plus one foliation of slope $e(x,y) \in \mathbf{C}\{x,y\}$. An algebraic web is presented (up to a change of coordinates) by an equation

$$F(x,y,y') = g \cdot P(y - y' \cdot x, y') = 0$$

where $P \in \mathbf{C}[s,t]$ is an affine equation of the reduced algebraic curve which defines the web, and g is invertible in $\mathbf{C}\{x,y\}$ (see for instance [14, Proposition 3.3]).

Among those webs, we aim to study exceptional ones. Exceptional webs occur in web geometry as special cases of maximal rank webs. They are d-webs which are not linearizable. This situation can only appear when $d \geqslant 5$, and classifying such webs remains a challenging problem in this geometry (see [13]). Currently known exceptional webs are of the preceding form, and particularly, the Bol 5-web.

Algebraic webs are linear ones, which means that the integral curves are (germs of) straight lines, so that the linearization polynomial is just 0 by Proposition 1. This allows us to give the following proposition:

Proposition 7. *Let $\mathcal{E}(d)$ be a d-web given by a $d-1$ algebraic web, plus one foliation of slope $e(x,y) \in \mathbf{C}\{x,y\}$. Such a web is presented by a differential equation*

$$\mathcal{E}(x,y,y') := P(y - y' \cdot x, y')\big(y' - e(x,y)\big) = 0,$$

where $P \in \mathbf{C}[s,t]$ is of multidegree $d-1$. Denote by $V_{\mathcal{E}}$ the linearization polynomial of $\mathcal{E}$, by R_P and $R_{\mathcal{E}}$ the resultant of P and $\mathcal{E}$ respectively.

Then

$$V_E = (-1)^{d-1} R_P \cdot P(y - ex, e) \cdot \big(\partial_x(e) + e\partial_y(e)\big) \cdot P(y - y'x, y').$$

Proof. One can check that, if p_i in $\mathbf{C}\{x,y\}$ for $1 \leqslant i \leqslant d$ denote the distincts roots of $\mathcal{E}$ in y', out of the singular locus $\{R_{\mathcal{E}}\} = 0$, then $V_{\mathcal{E}}(p_i) = (\partial_x(p_i) + p_i\partial_y(p_i)) \cdot R_{\mathcal{E}}$. Among those roots, suppose $e = p_d$. Then, the roots of $P(y - y'x, y')$ in y' are the p_i for $1 \leqslant i \leqslant d-1$.

Thus, the Lagrange interpolation formulae gives that

$$V_{\mathcal{E}} = \mathcal{E} \cdot R_{\mathcal{E}} \cdot \left(\frac{\partial_x(e) + e\partial_y(e)}{(y'-e)P(y-ex,e)} + \sum_{i=1}^{d-1} \frac{\partial_x(p_i) + p_i\partial_y(p_i)}{(y'-p_i)\prod_{j=1, j\neq i}^{d}(p_i - p_j)} \right).$$

Since the web defined by the previous equation is linear, one can show [2, Proposition 3] that $\partial_x(p_i) + p_i\partial_y(p_i) = 0$ for $1 \leqslant i \leqslant d-1$. By writing the resultant $R_{\mathcal{E}}$ thanks to the roots of $\mathcal{E}$, on can show that $R_{\mathcal{E}} = (-1)^{d-1} R_P \cdot P(y - ex, e)^2$. This gives the requested expression. $\square$

The implicit approach of webs *via* differential equations allows to prove (see [4, page 431]) that, for $d \geqslant 3$, the $\mathbf{C}$-vector space of abelian relations is isomorphic to the space of analytic solutions $(b_3, \ldots, b_d)$ of a non-homogeneous, linear differential system $\mathcal{M}(d)$:

$$\mathcal{M}(d)\begin{cases} \partial_x(b_d) + A_{1,1}\cdot b_3 + \cdots + A_{1,d-2}\cdot b_d = 0, \\ \partial_x(b_{d-1}) + \partial_y(b_d) + A_{2,1}\cdot b_3 + \cdots + A_{2,d-2}\cdot b_d = 0, \\ \qquad \vdots \\ \partial_x(b_3) + \partial_y(b_4) + A_{d-2,1}\cdot b_3 + \cdots + A_{d-2,d-2}\cdot b_d = 0, \\ \partial_y(b_3) + A_{d-1,1}\cdot b_3 + \cdots + A_{d-1,d-2}\cdot b_d = 0 \end{cases}$$

the coefficients $A_{i,j}$ being in $\mathcal{O}[1/R]$.

The main result of [4, Theorem 2] proves the existence of a $\mathbf{C}$-vector bundle E of rank π_d on $(\mathbf{C}^2, 0)$ which admits a connection

$$\nabla : E \longrightarrow \Omega^1 \otimes_{\mathcal{O}} E$$

such that the space $\operatorname{Ker}\nabla$ of its horizontal sections is isomorphic to the space of analytic solutions of the system $\mathcal{M}(d)$, and so, to $\mathcal{A}(d)$. Note that this connection, not necessarily integrable, is meromorphic with poles on the divisor defined by R.

To illustrate the important properties of the linearization polynomial in web geometry, we now give a direct implication of the preceding properties.

Proposition 8. *A* 5*-web $\mathcal{E}$ given by an algebraic* 4*-web and a foliation of slope $e(x, y) \in \mathbf{C}\{x, y\}$ of maximal rank is exceptional if and only if $\partial_x(e) + e\partial_y(e) \neq 0$, that is to say, the additional foliation is not a straight line.*

Proof. It is enough to check that under these assumptions, $V_{\mathcal{E}}$ is of degree less or equal to 3. Indeed, we know (see [14, Theorem 5.2] or [3, Théorème 2]) that if $\mathcal{W}(5)$ be a 5-web of maximal rank, then $\mathcal{W}(5)$ is linearizable if and only if its linearization polynomial is of degree less or equal to 3. We have $V_{\mathcal{E}} = R_P \cdot P(y - ex, e) \cdot (\partial_x(e) + e\partial_y(e)) \cdot P(y - y'x, y')$. Since e is not a root of $P(y - y'x, y')$, and P is of degree 4, we see that $V_{\mathcal{E}}$ is of degree at most 3 if and only if $\partial_x(e) + e\partial_y(e) = 0$. $\square$

Remark 5.1. The description of the linearization polynomial of $\mathcal{E}$ allows us to recover most of the exceptional 5-webs already known. More, using the connection, it gives a differential system, which expresses the maximality of the rank, that must satisfies e for $\mathcal{E}$ to be exceptional. It gives then a conceptual tool to classify these webs.

5.3. The connection associated with webs and the Cartan–Tresse linearization polynomial

Let $\mathcal{W}(d)$ be a d-web, presented by a differential polynomial P of order one and degree d, R denoting its resultant, with R_s an irreducible factor of R. We will assume that the leading coefficient a_0 of P is equal to 1 for more simplicity, but one can check that the results remain true in the general case. Let (E, ∇) be the associated connection. There exists an adapted basis B of E (meaning with "good properties," and which is defined in [14, Section 4.1]), which we fix now.

Theorem 2 allows to show the following theorem:

Theorem 6. *The coefficients A_{ij} in $\mathcal{M}(d)$, and thus, the connection, can only be written with the coefficients of U_P and V_P, defined by the relation of Theorem* 2.

Proof. See [14, Theorem 3.5]. □

Since we are interested in the study of non-singular webs, i.e. in a neighborhood of a point of $\mathbf{C}^2$ where R is different from 0, we will denote the well-defined quotient

$$\frac{V}{R} =: v_1 \cdot (y')^{d-1} + v_2 \cdot (y')^{d-2} + \cdots + v_d$$

where the coefficients v_i are in $\mathcal{O}[1/R]$.

For instance, when $d = 4$, the matrix connection can be written:

$$\gamma = \begin{pmatrix} A_1\,dx + (A_2 - v_2)\,dy & \xi_1 & \xi_2 \\ -dx & (A_1 - v_3)\,dx + (A_2 - v_2)\,dy & -v_1\,dy \\ -dy & v_4\,dx & A_1\,dx + A_2\,dy \end{pmatrix}$$

where $\xi_1 = (\partial_y(v_4) + v_4 v_2)\,dx + (v_1 v_4 + \partial_x(A_2 - v_2) - \partial_y(A_1 - v_3))\,dy$ $\xi_2 = (v_4 v_1 - (\partial_x(A_2) - \partial_y(A_1)))\,dx + (v_1 v_3 - \partial_x(v_1))\,dy$,

$$A_1 = -\frac{\partial_x(a_0)}{a_0} - \partial_y\left(\frac{a_1}{a_0}\right) - \frac{2a_2a_0 - a_1^2}{a_0^2}v_1 - \frac{a_1}{a_0}v_2 + 3v_3 \quad \text{and}$$
$$A_2 = -\frac{\partial_y(a_0)}{a_0} - \frac{a_1}{a_0}v_1 + 2v_2$$

It seems natural after studying the properties deduced from the fact that R divides V to see the corresponding results we can deduce on the connection associated with a web.

We are especially interested in the determinant of the connection $(\det E, \det\nabla)$. In the basis of $\det E$ deduced from the basis B of E, the connection $\det\nabla$ is the trace of the matrix γ of the connection ∇ (in B). We have the following expression of this trace:

Proposition 9. *The trace of the connection, in the basis B, is given by*:

$$\operatorname{tr}\gamma = \sum_{q=1}^{d-2}(d-1-q)(A_{q,d-1-q}\,dx + A_{d-q,q}\,dy).$$

Proof. See [14, Proposition 6.3]. □

Question. We go through a question asked by A. Hénaut about the nature of the poles of the determinant of the connection. Precisely, the question is to prove that, in case this determinant is integrable, the poles of the connection $(\det E, \det\nabla)$ are logarithmic. We recall that a meromorphic 1-form ω admits logarithmic poles along $\{R_s(x, y) = 0\} \subset \mathbf{C}^2$ if $R_s \cdot \omega$ and $R_s \cdot d\omega$ are holomorphic in a neighborhood of $\{R_s(x, y) = 0\}$.

We will denote by $ord_{R_s}(\det\nabla)$ the order of the poles of the meromorphic connection $\det\nabla$ on R_s in the basis of $\det E$ deduced from the basis B of E.

First, we can prove the following theorem:

Theorem 7. *Let P be a regular differential polynomial of order one in $K\{y\}$ of degree $d \geqslant 3$, its resultant being R and (E, ∇) the connection associated with the d-web defined by P. Assume that $rk_V(R_s) \geqslant 0$. Then, the order $ord_{R_s}(\det \nabla)$ of the poles of the meromorphic connection $(\det E, \det \nabla)$ on an irreducible factor R_s of R satisfies*

$$ord_{R_s}(\det \nabla) \leqslant rk_V(R_s).$$

Proof. Denote by U_k and V_k the polynomials of respective degrees $d-2$ and $d-1$ which satisfies the relations, for all $1 \leqslant i \leqslant d-3$:

$$(y')^k \cdot R \cdot \big(\partial_x(P) + y'\partial_y(P)\big) = U_k \cdot P + V_k \cdot S_P$$

with $U_0 = U$ and $V_0 = V$. They are obtained (see [14, Corollary 3.2]) the same way U_P and V_P are in Theorem 2. Lemma 3.2 in [14] asserts that the coefficients A_{ij} of $\mathcal{M}(d)$ are expressible in terms of the coefficients u_i^k and v_i^k of these polynomials divided by R:

$$A_{ij} = -\big(u_{d+1-i}^{d-2-j} + i \cdot v_{d-i}^{d-2-j}\big).$$

One can solve the coefficients of U with regard to the coefficients of V and P in the relation of Theorem 2. The corresponding expressions show that the multiplicity of R_s in U is at least the multiplicity of R_s in V, and so, the order of the poles of U/R and thus, of the coefficient $A_{i,d-2}$, is at most $rk_V(R_s)$.

Using the previous relations, one can show that we have the following polynomial equalities:

$$\begin{aligned}
&(V_k - pV_{k-1})(x, y, p) = -v_1^{k-1} \cdot R \cdot P(x, y, p) \quad \text{and} \\
&U_k - pU_{k-1} = v_1^{k-1} \cdot R \cdot \partial_p(P)(x, y, p)
\end{aligned}$$

where v_1^j is the leading coefficient of V_j/R. Then, we conclude, easily by induction, that the order of the poles of $A_{i,j}$, is at most $rk_V(R_s)$. □

Remark 5.2. If $rk_V(R_s) < 0$, we showed in [15, Proposition 11] that the connection (and so its determinant) is holomorphic.

Corollary 6. *Assume* $(\det E, \det \nabla)$ *is integrable. Then* $\det \nabla$ *has logarithmic poles on R_s if*

$$rk_V(R_s) \leqslant 1.$$

The hypothesis of integrability is crucial: the 3-web presented by $(y')^3 + xy^2(y')^2 + y^4(y') = 0$ is not of maximal rank, so the determinant of the associated connection (E, ∇) is not integrable. We can check that the order of the poles of $\det(\nabla)$ on y is 2, so that it has not logarithmic poles.

In [5, Proposition 1, Section 3], we can find the following definition for the notion of residue of a web:

Theorem 8. *Let $\mathcal{W}(d)$ be a d-web such that* $(\det E, \det\nabla)$ *is integrable. Let γ be a matrix of the connection ∇. The following conditions are equivalent*:

(i) $\operatorname{tr}(\gamma)$ *has logarithmic poles*;
(ii) *Up to a change of presentation of the web,*

$$\operatorname{tr}(\gamma) = \sum_q res_{R_q}\big[\mathcal{W}(d)\big] \cdot \frac{dR_q}{R_q}$$

where R_q is an irreducible factor of R and

$$res_{R_q}\big[\mathcal{W}(d)\big] := res_{R_q}\big[\operatorname{tr}(\gamma)\big] \in \mathbf{C}.$$

Proposition 10. *Under the hypothesis of Theorem* 8*, we assume that* $\operatorname{tr}(\gamma)$ *has logarithmic poles Then if $rk_V(R_s) \leqslant 0$ the residue of the web on R_s satisfies*

$$res_{R_s}\big[\mathcal{W}(d)\big] = 0.$$

Proof. By hypothesis, R_s divides the linearization polynomial V. Using the definition of the residue and the expression of the trace of the matrix of connection with V, we conclude that the residue must be nul on R_s. □

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